MATH2050C Selected Solutions to Assignment 5

Section 3.2 no. 14, 17, 18, 19, 21.

Section 3.3 no. 1, 3, 5, 7, 10, 11, 12, 13.

Section 3.2

(14b) Use Squeeze Theorem in $1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}$ and $\lim_{n\to\infty} n^{1/n} = 1$.

(19d) Use

$$\frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\frac{3}{n}\frac{4}{n}\cdots\frac{n}{n} \leq \frac{1}{n}\frac{2}{n}\frac{n}{n}\frac{n}{n}\cdots\frac{n}{n} = \frac{2}{n^2}.$$

By Squeeze Theorem we get

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \ .$$

(21) (a) The sequence $\{1/n\}$ converges to 0 and $\{(1/n)^{1/n}\}$ converges to 1.

(b) You can take $\{n\}$.

Note. In case $\lim_{n\to\infty} x_n^{1/n} = r \in (0,1)$, then $\lim_{n\to\infty} x_n = 0$. In case $r > 1$, then $\{x_n\}$ diverges to ∞. But there is no conclusion when $r = 1$. Both cases could happen.

Section 3.3

(5) $y_1 = \sqrt{p}, p > 0$, and $y_{n+1} = \sqrt{p + y_n}$. Use induction it is straightforward to see $\{y_n\}$ is increasing. To show boundedness we follow the hint and use induction to show $y_n \leq 1 + 2\sqrt{p}$. Assuming $y_n \leq 1 + 2\sqrt{p}$, we have

$$y_{n+1}^2 = p + y_n \leq p + 1 + 2\sqrt{p} = (1 + \sqrt{p})^2,$$

hence

$$y_{n+1} \leq 1 + \sqrt{p} < 1 + 2\sqrt{p} \ .$$

(7) It is clear that $x_{n+1} = x_n + 1/x_n, x_1 > 0$, is increasing. Were it bounded from above, its limit exists by Monotone Convergence Theorem. Letting the limit be $b > 0$, then passing limit in the defining relation of the sequence we get $b = b + 1/b$, which is ridiculous. We conclude that $\{x_n\}$ is divergent to infinity.

(10). We claim the sequence $\{y_n\}$ given by

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n},$$

is increasing and bounded. First, we have

$$y_n < \frac{1}{n} + \frac{1}{n} + \cdots \frac{n}{n} = \frac{n}{n} = 1 \ , \quad \forall n \geq 1$$

, hence $\{y_n\}$ is bounded from above. Next,

$$y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \ .$$

We have

$$y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > 0 \ , \quad \forall n \geq 1 \ ,$$

hence it is increasing. By Monotone Convergence Theorem $\{y_n\}$ is convergent.

Note. One can show that the limit is $\log 2$.

(11) We claim that the sequence $x_n = \sum_{k=1}^{n} \dfrac{1}{k^2}$ is convergent. It is bounded from above:

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2 \ .$$

Since it is clearly increasing, by Monotone Convergence Theorem $\{x_n\}$ is convergent.

Note. As discovered by Euler, the limit is $\pi^2/6$.

(12)(a) By Limit Theorem

$$\lim_{n\to\infty} = \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \ .$$

(b) By Limit Theorem

$$\lim_{n\to\infty} = \left(1 + \frac{1}{n}\right)^{2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^2 \ .$$

(c) By Limit Theorem

$$\lim_{n\to\infty} = \left(1 + \frac{1}{n+1}\right)^n = \frac{\lim_{n\to\infty} \left(1 + \dfrac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty} \left(1 + \dfrac{1}{n+1}\right)} = e \ .$$

(d) We have

$$1 - \frac{1}{n} = \frac{1 - \dfrac{1}{n^2}}{1 + \dfrac{1}{n}} \ .$$

If we can show that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n = 1, \quad (*)$$

then by Limit Theorem and the above expression,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{\lim_{n\to\infty} (1 - 1/n^2)^n)}{\lim_{n\to\infty} (1 + 1/n)^n} = \frac{1}{e} \ .$$

To prove (*),

$$\left(1-\frac{1}{n^2}\right)^n = 1+\sum_{k=1}^{n}\binom{n}{k}\left(\frac{-1}{n}\right)^{2k} \ .$$

We estimate the general term by

$$\begin{aligned}\left|\binom{n}{k}\left(\frac{-1}{n}\right)^{2k}\right| &= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{1}{n^{2k}} \\ &= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{n^k} \\ &\le \frac{1}{n^k}\ , \quad \forall k\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\left(1-\frac{1}{n^2}\right)^n - 1\right| &\le \frac{1}{n^2}+\left|\sum_{k=2}^{n}\binom{n}{k}\left(\frac{-1}{n}\right)^{2k}\right| \\ &\le \frac{1}{n^2}+\sum_{k=2}^{n}\frac{1}{n^k} \\ &\le \frac{1}{n^2}+\frac{n}{n^2} \\ &\le \frac{2}{n} \to 0, \quad \text{as } n\to\infty\ .\end{aligned}$$